Swelling dynamics of constrained thin-plate gels under an external force

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We analyze the swelling kinetics of constrained thin-plate gels using the linearized stress-diffusion coupling model proposed in the previous paper [T. Yamaue and M. Doi, Phys. Rev. E **69**, 041402 (2004)]. The gel is chemically clamped on the disklike glass plates at the top and the bottom surfaces and can swell and shrink only along the thickness direction. We analyze how the top plate moves when a force is applied at a certain point on the top plate while the bottom plate is fixed. We predict that (i) the translation and the rotation of the top plate are described by a single exponential relaxation process, that (ii) the rotational relaxation process is three times faster than the translational one, and that (iii) when the force is applied to the edge, the displacement by the rotation is four times larger than that by the translation at the edge point where the force is applied. We also analyze how the gel deforms when it is clamped on the flexible film on which external load is applied.

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I. INTRODUCTION

A gel placed in solution absorbs (or desorbs) the solutions and swells (or shrinks) when an external force is applied on it. The phenomenon is important in various industrial processes such as coating and printing [2]. It is also important in the study of gels as actuators and sensors [3,4]. Although the process has been known for a long time, proper theoretical framework to analyze the process has not been given.

In the previous paper [1], we have shown that the experiments done by Suzuki *et al.* of the swelling of a thin plate gel whose two surfaces are chemically clamped to the glass plates [5] cannot be analyzed by the classical model, which describes the swelling kinetics of a spherical gel [6,7]. We have formulated the dynamics of gels using the linearized stress-diffusion coupling model [8,9] and have shown that the experiments of Suzuki *et al.* are naturally explained by this model [1].

In this paper, we extend the previous work and consider the swelling kinetics of the disklike gel under external forces. The situation we consider is explained in Fig. 1. In this system, the top and bottom surfaces of gels are chemically clamped on the glass plates, and the gels can swell and shrink only along the thickness direction. We analyze the time evolution of the thickness and the pressure when a force is applied to a point on the top glass plate. We also discuss the swelling kinetics of a disklike gel clamped to a flexible surface (such as thin polymer films).

II. THE LINEARIZED STRESS DIFFUSION COUPLING MODEL FOR A THIN-PLATE GEL

We first summarize the linearized stress-diffusion coupling model for gels [1]. Let $\mathbf{u}(\mathbf{r},t)$ be the displacement of a point located at $\mathbf{r}$ in the reference state and $\dot{\mathbf{u}}(\mathbf{r},t)$ be its time derivative $[\dot{\mathbf{u}}(\mathbf{r},t)=\partial\mathbf{u}(\mathbf{r},t)/\partial t]$. Let $\mathbf{v}_s(\mathbf{r},t)$ be the velocity of the solvent. The equations of motion which determine $\mathbf{u}(\mathbf{r},t)$ and $\mathbf{v}_s(\mathbf{r},t)$ are as follows.

$$\mathbf{v}_s - \dot{\mathbf{u}} = -\frac{1-\phi}{\zeta}\nabla p, \qquad (1)$$

$$\nabla\cdot\boldsymbol{\sigma} = \nabla p, \qquad (2)$$

$$\phi\nabla\cdot\dot{\mathbf{u}} + (1-\phi)\nabla\cdot\mathbf{v}_s = 0. \qquad (3)$$

Here, ζ is the friction constant associated with the motion of the polymer relative to the solvent, ϕ is the volume fraction of polymer, and p is the pressure. The first equation represents Darcy's law for the permeation of solvent through the gel network. The second equation stands for the force balance, where $\boldsymbol{\sigma}$ is the stress of the gel network. The third equation stands for the incompressibility condition.

The stress $\boldsymbol{\sigma}$ is given by the constitutive equation for the gel network. Here, we use the linearized form for the stress tensor:

$$\sigma_{ij} = K\sum_k \frac{\partial u_k}{\partial x_k}\delta_{ij} + G\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\sum_k \frac{\partial u_k}{\partial x_k}\delta_{ij}\right), \qquad (4)$$

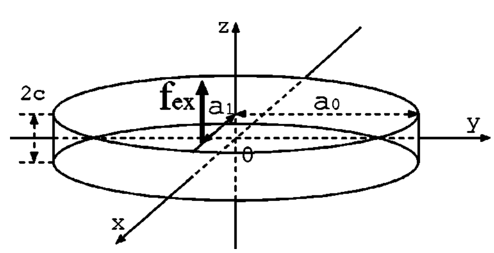

FIG. 1. Model of the thin-plate gel with rigid surfaces applied with an external force.

where K is the bulk modulus and G is the shear modulus of gels. Eqs. (2) and (4) give

$$\left(K+\frac{G}{3}\right)\nabla(\nabla\cdot\mathbf{u})+G\nabla^2\mathbf{u}=\nabla p. \tag{5}$$

By eliminating the velocity of solvent from Eqs. (1) and (3), we have

$$\nabla\cdot\dot{\mathbf{u}}=\frac{(1-\phi)^2}{\zeta}\nabla^2 p. \tag{6}$$

Equations (5) and (6) are the closed set which determine $\mathbf{u}(\mathbf{r},t)$ and $p(\mathbf{r},t)$.

For the disklike gel shown in Fig. 1, we take the origin of the cylindrical coordinate at the center point of the gel. Let a_0 be the initial radius of the gel and $2c_0$ be the initial thickness of the gel. We assume that the thickness of the gel is much smaller than the radius of the gel ($c_0 \ll a_0$).

In this system, since the top and bottom surfaces of the gel are chemically clamped on the plates, we can assume that the displacement vector (u_r,u_θ,u_z) is of the same order of the thickness of the gel $c(t)$: $u_r,u_\theta,u_z \sim c(t)$. Therefore the terms of $\partial_r u_i$ and $r^{-1}\partial_\theta u_i$ are much smaller than those of $\partial_z u_i$. In such a case, Eq. (5) can be approximated as

$$\begin{aligned}\frac{\partial p}{\partial r}&=\left(K+\frac{4}{3}G\right)\left(\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}\right)+G\left(\frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta^2}+\frac{\partial^2 u_r}{\partial z^2}\right)\\&\quad+\left(K+\frac{G}{3}\right)\left(-\frac{u_r}{r^2}+\frac{1}{r}\frac{\partial^2 u_\theta}{\partial r\,\partial\theta}-\frac{1}{r^2}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial^2 u_z}{\partial r\,\partial z}\right)\\&\cong G\frac{\partial^2 u_r}{\partial z^2},\end{aligned} \tag{7}$$

$$\begin{aligned}\frac{1}{r}\frac{\partial p}{\partial\theta}&=\left(K+\frac{4}{3}G\right)\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2}+G\left(\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}+\frac{\partial^2 u_\theta}{\partial z^2}\right)\\&\quad+\left(K+\frac{G}{3}\right)\left(\frac{1}{r}\frac{\partial^2 u_r}{\partial r\,\partial\theta}+\frac{1}{r^2}\frac{\partial u_r}{\partial\theta}+\frac{1}{r}\frac{\partial^2 u_z}{\partial\theta\,\partial z}\right)\cong G\frac{\partial^2 u_\theta}{\partial z^2},\end{aligned} \tag{8}$$

$$\begin{aligned}0&=\left(K+\frac{4}{3}G\right)\frac{\partial^2 u_z}{\partial z^2}+G\left(\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial\theta^2}\right)+\left(K+\frac{G}{3}\right)\\&\quad\times\left(\frac{\partial^2 u_r}{\partial r\,\partial z}+\frac{1}{r}\frac{\partial u_r}{\partial z}+\frac{1}{r}\frac{\partial^2 u_\theta}{\partial\theta\,\partial z}\right)\cong\left(K+\frac{4}{3}G\right)\frac{\partial^2 u_z}{\partial z^2}.\end{aligned} \tag{9}$$

Here, we have assumed that the pressure $p(r,\theta,z,t)$ is independent of z, since the equilibration time of the pressure in the z direction is of the order of c_0^2/D, where D is the collective diffusion constant, and is negligibly small compared with the characteristic time (which is a_0^2/D) of the swelling.

Since $|\partial\dot{u}_r/\partial r|$ and $|r^{-1}\partial\dot{u}_\theta/\partial\theta|$ are much smaller than $|\partial\dot{u}_z/\partial z|$, and p is independent of z, Eq. (6) can be approximated by

$$\frac{\partial\dot{u}_z}{\partial z}=\frac{(1-\phi)^2}{\zeta}\left(\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}+\frac{1}{r^2}\frac{\partial^2 p}{\partial\theta^2}\right). \tag{10}$$

The boundary condition is described as

$$p(r=a_0,\theta,t)=0 \text{ for the boundary at the side}. \tag{11}$$

From Eq. (9), we have

$$u_z=A(r,\theta,t)z. \tag{12}$$

From Eqs. (7) and (8), the displacements u_r and u_θ are described as follows.

$$u_r(r,\theta,z,t)=\frac{1}{2G}\frac{\partial p}{\partial r}(r,\theta,t)(c_0^2-z^2), \tag{13}$$

$$u_\theta(r,\theta,z,t)=\frac{1}{2G}\frac{1}{r}\frac{\partial p}{\partial\theta}(r,\theta,t)(c_0^2-z^2). \tag{14}$$

Here, we have used the boundary conditions that the gel is clamped at the two surfaces, which are described as

$$u_r(r,\theta,z=\pm c,t)=0 \tag{15}$$

$$u_\theta(r,\theta,z=\pm c,t)=0 \text{ for the top and bottom boundary}.$$

By Eq. (12), Eq. (10) is rewritten as

$$\dot{A}(r,\theta,t)=\frac{(1-\phi)^2}{\zeta}\left(\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}+\frac{1}{r^2}\frac{\partial^2 p}{\partial\theta^2}\right). \tag{16}$$

In order to solve this equation, we use the boundary condition at the surface.

When the top and bottom surfaces of a gel are clamped on rigid surfaces, such as glass plates, the swelling ratio $A(r,\theta,t)$ is generally described by six time-dependent amplitudes, which correspond to six degrees of freedom of the top glass plate. The time evolution of each amplitude is calculated by the six conditions of the mechanical balance and the momentum balance of the rigid surface described as

$$\int_0^{a_0}dr\int_0^{2\pi}d\theta\, r(\boldsymbol{\sigma}-p\mathbf{I})_{z=c}\cdot\mathbf{n}_z=\mathbf{f}, \tag{17}$$

$$\int_0^{a_0}dr\int_0^{2\pi}d\theta\, r[(\boldsymbol{\sigma}-p\mathbf{I})_{z=c}\cdot\mathbf{n}_z]\times\mathbf{r}=\mathbf{f}\times\mathbf{r}, \tag{18}$$

where $\mathbf{f}$ is a force acting on a point of the rigid plate.

When the top and bottom surfaces of a gel are clamped on flexible impermeable membranes, the condition of the mechanical balance in the z direction on any point of the flexible membrane is described as

$$\sigma_{zz}(r,\theta,t)-p(r,\theta,t)=f_z(r,\theta), \tag{19}$$

where $f_z(r,\theta)$ is the external force acting in the z direction on a point at (r,θ,c_0) of the flexible surface. Since $|\partial u_r/\partial r|$ and $|r^{-1}\partial u_\theta/\partial\theta|$ are much smaller than $|\partial u_z/\partial z|$, from Eq. (12), σ_{zz} is related with $A(r,\theta,t)$ as follows:

$$\sigma_{zz} = \left(K + \frac{4}{3}G\right)\frac{\partial u_z}{\partial z} + \left(K - \frac{2}{3}G\right)\left(\frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right)$$
$$\cong \left(K + \frac{4}{3}G\right)A(r,\theta,t). \quad (20)$$

From Eq. (19), Eq. (16) is rewritten as the diffusion equation for the pressure.

$$\frac{\partial p}{\partial t} = D\left(\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2}\right), \quad (21)$$

where D is the collective diffusion constant of gels defined as $D \equiv (1-\phi)^2(K+4G/3)/\zeta$. Equation (21) can be solved for p since the boundary condition for p is known. This diffusion equation leads to a multimode relaxation process of a thin-plate gel with a boundary condition of Eq. (11).

III. SWELLING BY A FORCE ACTING ON THE GLASS PLATE

We consider the situation that the external force $\mathbf{f}_{\text{ex}} = (0,0,f_{\text{ex}})$ is applied to a point on the top glass plate, $(r,\theta,z) = (a_1,0,c_0)$, in the direction normal to the top surface (see Fig. 1). Our objective is to find out the time dependence of the displacement $c(r,\theta,t)$ of the top surface.

Since the surface must represent a plane, $A(r,\theta,t)$ must be written as

$$A(r,\theta,t) = \alpha_0(t) + \alpha_1(t) r \cos\theta, \quad (22)$$

where $\alpha_0(t)$ and $\alpha_1(t)$ are the functions of time to be determined. Equations (16) and (22) lead to the following Poisson's equation for the pressure $p(r,\theta,t)$.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} = \beta_0(t) + \beta_1(t) r \cos\theta, \quad (23)$$

where

$$\beta_0(t) \equiv \frac{\zeta}{(1-\phi)^2}\dot{\alpha}_0(t), \quad (24)$$

$$\beta_1(t) \equiv \frac{\zeta}{(1-\phi)^2}\dot{\alpha}_1(t). \quad (25)$$

Now, we assume that the pressure p depends on θ as follows:

$$p(r,\theta,t) \equiv f(r,t) + g(r,t)\cos\theta. \quad (26)$$

Equations (23) and (26) lead to the following equations of $f(r,t)$ and $g(r,t)$:

$$\beta_0(t) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right), \quad (27)$$

$$\beta_1(t) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial g}{\partial r}\right) - \frac{g}{r^2}. \quad (28)$$

These equations can be solved under the boundary condition of Eq. (11), which is rewritten as $f(a_0,t)=0$ and $g(a_0,t)=0$:

$$f(r,t) = \frac{1}{4}\beta_0(t)(r^2 - a_0^2), \quad (29)$$

$$g(r,t) = \frac{1}{8}\beta_1(t) r (r^2 - a_0^2). \quad (30)$$

The z component of the total force acting on the top glass plate must balance with the external force f_{ex}.

$$\int_0^{a_0} dr \int_0^{2\pi} d\theta\, r(\sigma_{zz} - p)\big|_{z=c} = f_{\text{ex}}. \quad (31)$$

Furthermore, the torque balance equation for the top glass plate is written as

$$\int_0^{a_0} dr \int_0^{2\pi} d\theta\, r^2 \cos\theta(\sigma_{zz} - p)\big|_{z=c} = f_{\text{ex}} a_1. \quad (32)$$

From Eqs. (20) and (22), σ_{zz} is given by

$$\sigma_{zz} \cong \left(K + \frac{4}{3}G\right)[\alpha_0(t) + \alpha_1(t) r \cos\theta]. \quad (33)$$

From Eqs. (26), (29), (30), and (33), Eqs. (31) and (32) are written as

$$\left(K + \frac{4}{3}G\right)\alpha_0(t) = \frac{f_{\text{ex}}}{\pi a_0^2} - \frac{a_0^2}{8}\beta_0(t), \quad (34)$$

$$\left(K + \frac{4}{3}G\right)\alpha_1(t) = \frac{4 f_{\text{ex}} a_1}{\pi a_0^4} - \frac{a_0^2}{24}\beta_1(t). \quad (35)$$

Equations (24) and (34) give the following equation for the translation of the top plate.

$$\tau_t \dot{\alpha}_0(t) = \frac{f_{\text{ex}}}{\pi a_0^2}\left(K + \frac{4}{3}G\right)^{-1} - \alpha_0(t), \quad (36)$$

where τ_t is the relaxation time for the translation defined by

$$\tau_t \equiv \frac{a_0^2}{8}D^{-1}. \quad (37)$$

Equations (25) and (35) give the following equation for the rotation of the top plate:

$$\tau_r \dot{\alpha}_1(t) = \frac{4 f_{\text{ex}} a_1}{\pi a_0^4}\left(K + \frac{4}{3}G\right)^{-1} - \alpha_1(t), \quad (38)$$

where τ_r is the relaxation time for the rotation defined by

$$\tau_r \equiv \frac{a_0^2}{24}D^{-1}. \quad (39)$$

From Eqs. (36) and (38), we can solve $\alpha_0(t)$ and $\alpha_1(t)$ as follows:

$$\alpha_0(t) = \frac{f_{\text{ex}}}{\pi a_0^2}\left(K + \frac{4}{3}G\right)^{-1}\left\{1 - \exp\left(-\frac{t}{\tau_t}\right)\right\}, \quad (40)$$

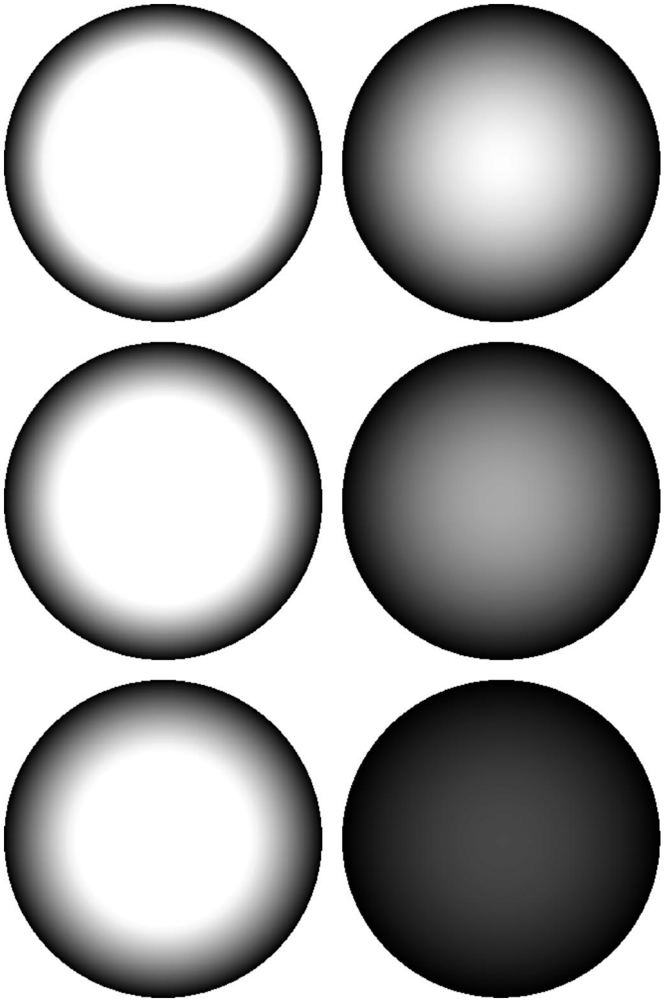

FIG. 2. The time evolution of the pressure p in gels, where an external force is applied to the center of the top glass plate $[-2f_{\mathrm{ex}}/(\pi a_0^2)\!:\!\text{white}\rightarrow 0.0\!:\!\text{black}]$, at the time of $t/\tau_{\mathrm{t}}=0, 0.2, 0.4, 0.8, 1.2$, and 2.0 (from left to right).

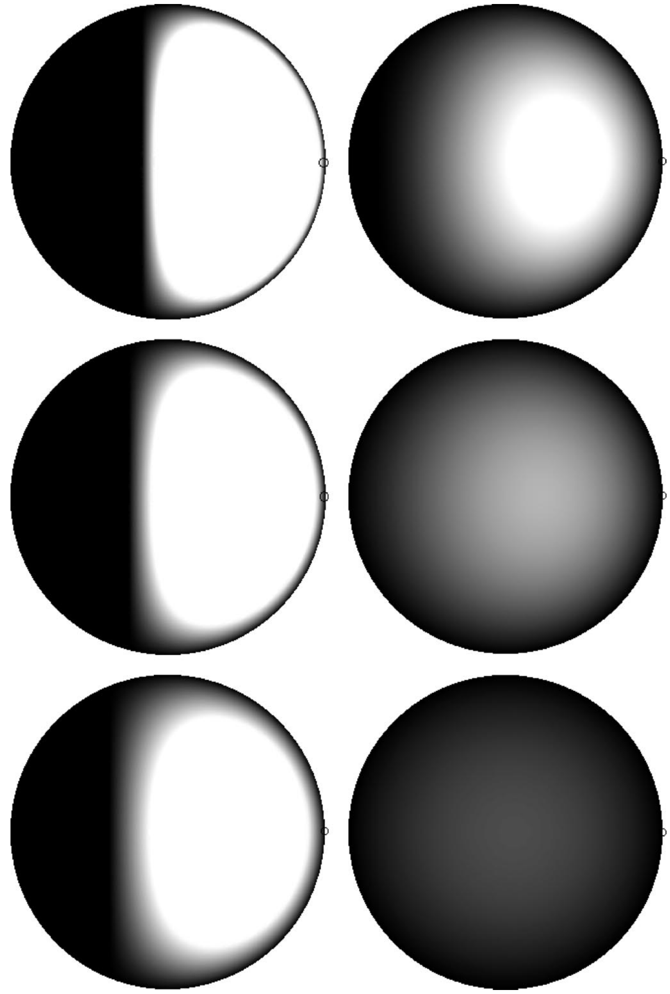

FIG. 3. The time evolution of the pressure p in gels, where an external force is applied to a right edge point marked by a small circle of the top glass plate $[-2f_{\mathrm{ex}}/(\pi a_0^2)\!:\!\text{white}\rightarrow 0.0\!:\!\text{black}]$, at the time of $t/\tau_{\mathrm{t}}=0, 0.2, 0.4, 0.8, 1.2$, and 2.0 (from left to right).

$$\alpha_1(t) = \frac{4f_{\mathrm{ex}}}{\pi a_0^3}\frac{a_1}{a_0}\left(K+\frac{4}{3}G\right)^{-1}\left\{1-\exp\left(-\frac{t}{\tau_{\mathrm{r}}}\right)\right\}. \quad (41)$$

Here, we used the initial conditions $\alpha_0(0)=0$ and $\alpha_1(0)=0$. Therefore Eq. (22) leads to

$$u_{\mathrm{z}}(r,\theta,z,t) = \frac{f_{\mathrm{ex}}}{\pi a_0^2}\left(K+\frac{4}{3}G\right)^{-1}\left[1-\exp\left(-\frac{t}{\tau_{\mathrm{t}}}\right) + 4\frac{a_1}{a_0}\left\{1-\exp\left(-\frac{t}{\tau_{\mathrm{r}}}\right)\right\}\frac{r}{a_0}\cos\,\theta\right]z. \quad (42)$$

The time evolution of the thickness of gels $c(r,\theta,t)$ is obtained as follows:

$$\frac{c(r,\theta,t)}{c_0} = 1 + \frac{f_{\mathrm{ex}}}{\pi a_0^2}\left(K+\frac{4}{3}G\right)^{-1}\left[1-\exp\left(-\frac{t}{\tau_{\mathrm{t}}}\right) + 4\frac{a_1}{a_0}\left\{1-\exp\left(-\frac{t}{\tau_{\mathrm{r}}}\right)\right\}\frac{r}{a_0}\cos\,\theta\right]. \quad (43)$$

These results show the following three predictions: (i) the time evolution of the thickness of gels is described by the superposition of a translation and a rotation, each of which is described by a single exponential and the characteristic relaxation time depends on the radius of the circular surface, (ii) the relaxation time of the rotation is one-third of that of the translation, and (iii) the amplitude of the displacement of the top surface due to the rotation is $4a_1/a_0$ times larger than that due to the translation at the edge $(r=a_0)$.

From Eqs. (26), (29), (30), (34), and (35), the time evolution of the pressure in the gel is solved as follows:

$$p(r,\theta,t) = \frac{2f_{\mathrm{ex}}}{\pi a_0^2}\left\{\exp\left(-\frac{t}{\tau_{\mathrm{t}}}\right) + 6\frac{a_1}{a_0}\exp\left(-\frac{t}{\tau_{\mathrm{r}}}\right)\frac{r}{a_0}\cos\,\theta\right\} \times\left\{\left(\frac{r}{a_0}\right)^2 - 1\right\}. \quad (44)$$

Figure 2 shows the time evolution of the pressure when an external force is acting on the center of the glass plate $(a_1=0)$, at the time $t/\tau_{\mathrm{t}}=0, 0.2, 0.4, 0.8, 1.2$, and 2.0. The profiles of the time evolution of the pressure are invariant during

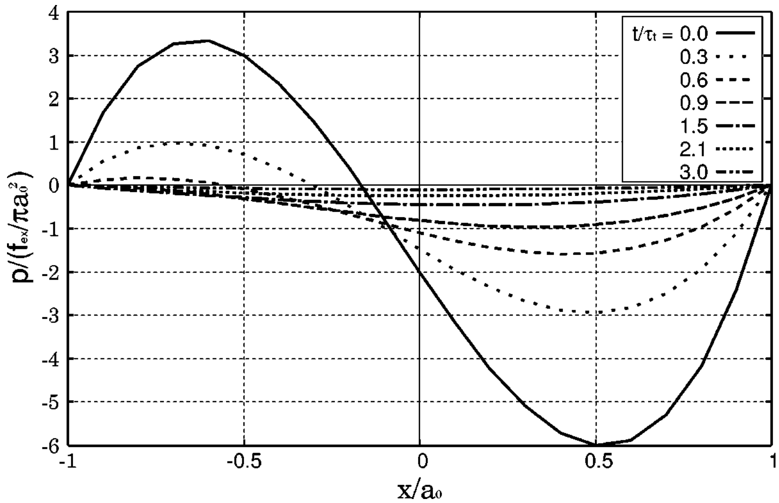

FIG. 4. The profiles of the nondimensional pressure $p\pi a_0^2/f_{ex}$ in gels in the section along the x axis, where an external force is applied to a point on the edge of the top glass plate, at the time of $t/\tau_t = 0, 0.3, 0.6, 0.9, 1.5, 2.1$, and 3.0.

the swelling process, and that is one of the special properties of the diffusion in a single exponential swelling process. This property agrees with that of the swelling process of a thin-plate gel due to changing of the osmotic pressure of the solvent [1].

On the other hand, when an external force is applied to a point on the edge of the glass plate ($a_1 = a_0$), the time evolution of the pressure becomes as it is shown in Fig. 3. The time evolution of the pressure profiles in the section along the x axis is shown in Fig. 4. We can see that a large pressure gradient is created in the x direction in the beginning, which causes the solvent flux in the x direction and the rotation of the top glass plate. The large pressure gradient is relaxed rapidly with the rotation of the plate, whose relaxation time is $\tau_r (= \tau_t/3)$, and until the time of $t \sim \tau_t$, the rotation is almost finished and the profile of the the pressure becomes to be symmetric in the center of the plate.

Figure 5 shows the time evolution of the displacement

$$\left(\frac{c(r,\theta,t)}{c_0} - 1\right)\frac{\pi a_0^2}{f_{ex}}\left(K + \frac{4}{3}G\right) \tag{45}$$

calculated at the three points on the top plate, $(r,\theta,z) = (a_0, 0, c_0), (0,0,c_0)$, and (a_0, π, c_0). From Fig. 5 we can confirm that the relaxation time of the rotation τ_r is faster than that of the translation τ_t, and the rotation is finished rapidly in the time of $t \sim \tau_t$. After that, the gel swells by the parallel translation of the top glass plate in the z direction while keeping the rotation angle. The amplitude of the displacement due to the rotation on the edge point is four times larger than that due to the translation.

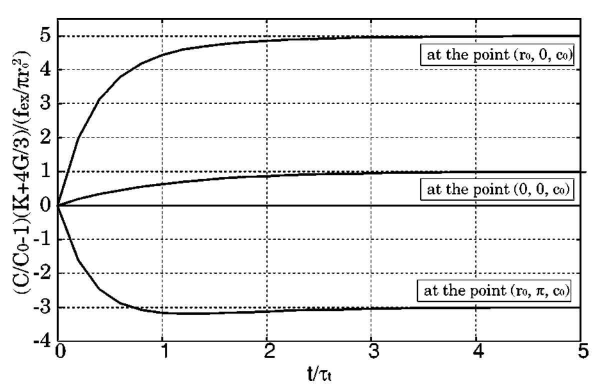

FIG. 5. The time evolution of the nondimensional displacement $[c(r,\theta,t)/c_0 - 1](K + 4G/3)\pi a_0^2/f_{ex}$ of the three points on the top plate, where the upper line shows the displacement of the point on the edge: $(r,\theta,z) = (a_0, 0, c_0)$, the middle line shows that of the center point: $(r,\theta,z) = (0,0,c_0)$, and the lower line shows that of the point on the edge: $(r,\theta,z) = (a_0, \pi, c_0)$.

IV. SWELLING BY A FORCE ACTING ON THE FLEXIBLE MEMBRANE

Next, we consider how the swelling behavior changes if the top and bottom surfaces of a gel are clamped on flexible impermeable membranes. Figure 6 shows the situation we consider. An external force $\mathbf{f}_{ex} = (0, 0, f_{ex})$ is applied on a circular region on the top membrane, whose radius is a_1, while the bottom memblane is fixed. Our objective is to find out the time dependence of the displacement $c(r,t)$ of the top flexible membrane.

From Eqs. (19) and (20), the mechanical balance in the z direction on any point of the flexible membrane is described as

$$\left(K + \frac{4}{3}G\right)A(r,t) - p(r,t) = \frac{f_{ex}}{\pi a_1^2}\Theta(a_1 - r). \tag{46}$$

Here, A and p do not depend on θ. The analytic solution of this diffusion equation, Eq. (21), is written as

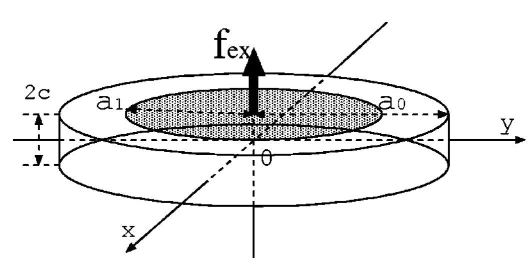

FIG. 6. Model of the thin-plate gel with a flexible impermeable membrane applied with an external force.

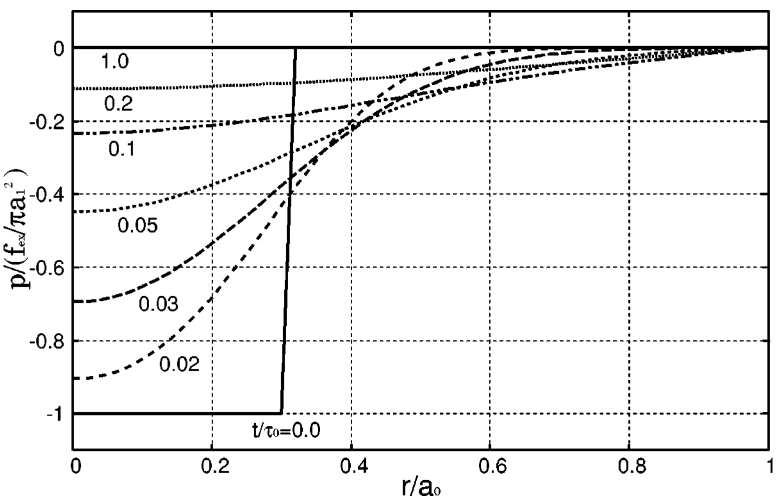

FIG. 7. The time evolution of the pressure $p(r,t)$ for a swelling process of a gel clamped on impermeable membranes under a force applied on a circle region of the radius $a_1=0.3a_0$ at the time of $t/\tau_0=0,0.02,0.03,0.05,0.1,0.2$, and 1.0.

$$p(r,t)=\sum_{n=0}^{\infty}C_n J_0\left(\sqrt{\frac{\lambda_n}{D}}r\right)\exp(-\lambda_n t), \tag{47}$$

where $J_0(x)$ is the zero order function of the first kind Bessel's functions. The multimode relaxation times λ_n are given by $\lambda_n=\tau_0\alpha_n$ for the α_n calculated by $J_0(\alpha_n)=0$, which is the boundary condition of Eq. (11). Here, the characteristic relaxation time τ_0 is defined by $\tau_0\equiv D/a_0^2$. The initial condition is described as

$$p(r,t=0)=-\frac{f_{ex}}{\pi a_1^2}\Theta(a_1-r), \tag{48}$$

which is lead by the condition $A(r,t=0)=0$. From the initial condition Eq. (48), the coefficients C_n are calculated as

$$C_n=-\frac{2f_{ex}}{\pi a_0^2 a_1^2}\left[J_1\left(\sqrt{\frac{\lambda_n}{D}}a_0\right)\right]^{-2}\int_0^{a_1}dr\ rJ_0\left(\sqrt{\frac{\lambda_n}{D}}r\right), \tag{49}$$

where $J_1(x)$ denotes the first order function of the first kind Bessel's functions.

Figure 7 shows the time evolution of the pressure $p(r,t)$ calculated numerically for $a_1=0.3a_0$, and the time evolution of the displacement calculated by the pressure $p(r,t)$ and Eq. (46) is shown in Fig. 8.

The swelling process of a thin-plate gel for the gel clamped on flexible impermeable membranes becomes a multimode relaxation process.

V. SUMMARY

In this paper, we have calculated the swelling process of thin-plate gels with circular surfaces under an external force. At first, we analyze the time evolution of the thickness and the pressure, when a force is applied to a point on the top glass plate. The results show that the displacement of the top surface is described by both the translation and the rotation and each of them is described by a single exponential relaxation process. Here, we show the relaxation of rotation is three times faster than that of translation, and the displacement by rotation is four times larger than that by translation at the point on the edge, when a force is applied to a point on the edge.

We also analyze the time evolution of the thickness and the pressure, when a gel is clamped on the flexible impermeable film on which external load is applied. The results show

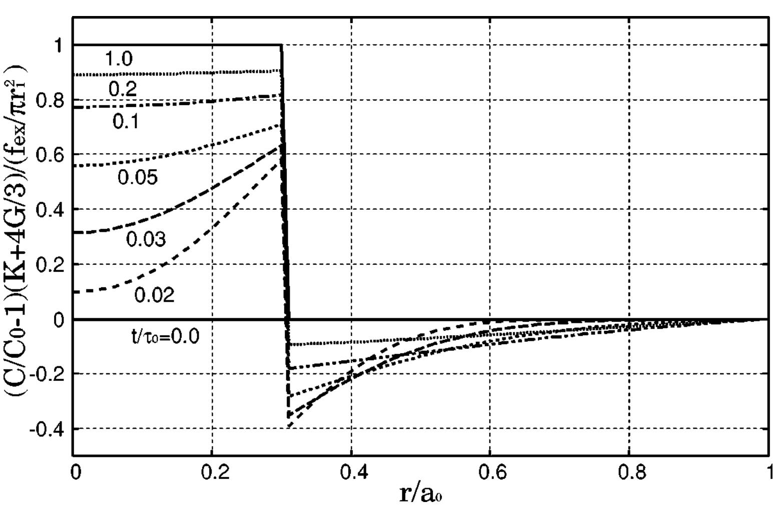

FIG. 8. The time evolution of the nondimensional displacement $[c(r,t)/c_0-1](K+4G/3)\pi a_1^2/f_{ex}$ for a swelling process of a gel clamped on impermeable membranes under a force applied on a circle region of the radius $a_1=0.3a_0$ at the time of $t/\tau_0=0,0.02,0.03,0.05,0.1,0.2$, and 1.0.

that the swelling process becomes a multimode relaxation process.

These results show that the stress-diffusion coupling model of gels, which considers the relative motion between the solvent and the polymer of gels together with the mechanical coupling between the solvent diffusion and the polymer network stress, can describe an unusual feature for the solvent permeation kinetics of a gel under external forces.

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